The nonlinear Schrödinger equation on metric graphs

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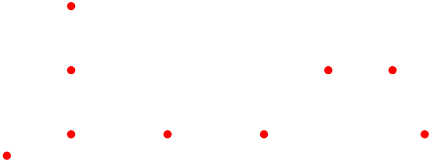
Friday 20 September 2024

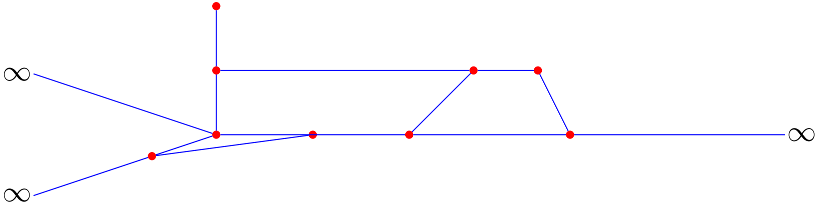
What is a metric graph?

A metric graph is made of vertices



What is a metric graph?

A metric graph is made of vertices and of edges joining the vertices or going to infinity.



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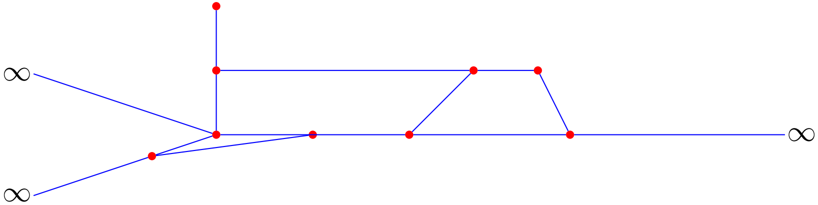

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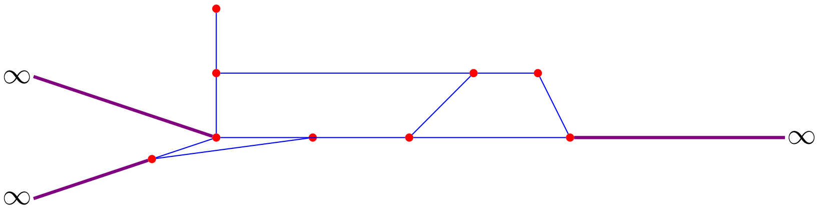

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- the edges going to infinity are halflines and have *infinite length*.

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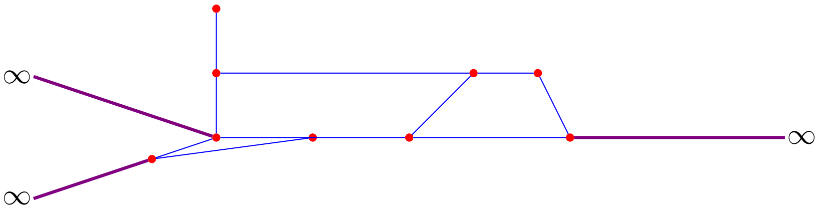
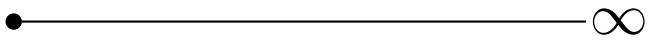


- *metric* graphs: the lengths of edges are important.
- the edges going to infinity are halflines and have *infinite length*.
- a metric graph is *compact* if and only if it has a finite number of edges of finite length.

Constructions based on halflines



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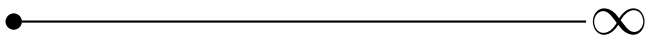

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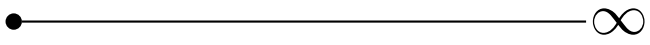

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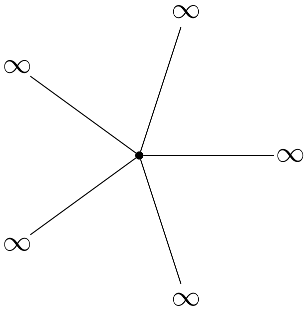

The 5-star graph

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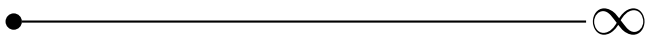

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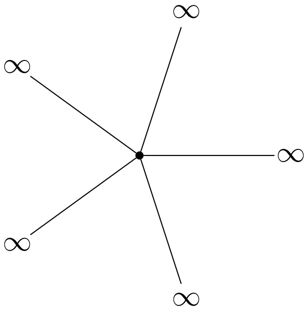

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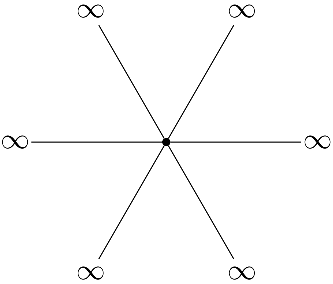
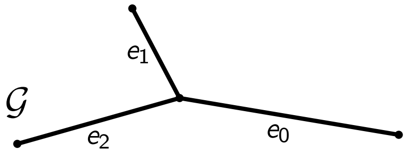


The 6-star graph

Functions defined on metric graphs



A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3)

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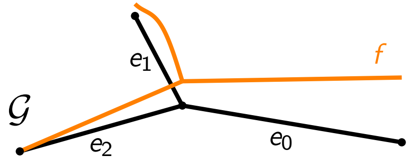

A metric graph $\mathcal{G}$ with three edges e_0 (length 5), e_1 (length 4) and e_2 (length 3), a function $f : \mathcal{G} \to \mathbb{R}$

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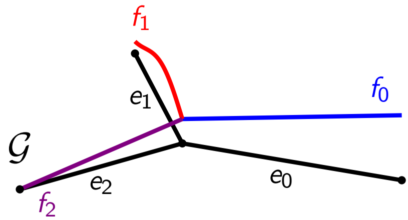

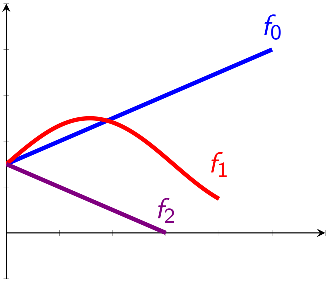

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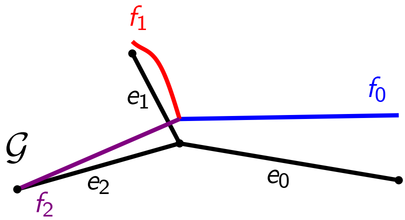

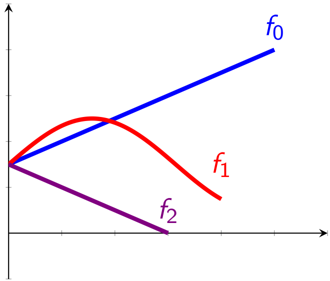

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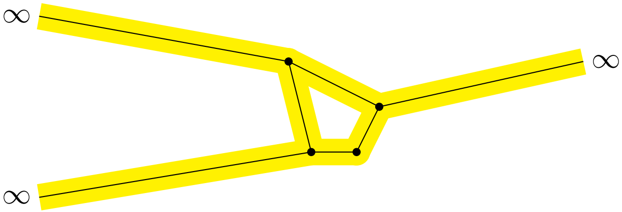
$$\int_{\mathcal{G}} f \, \mathrm{d}x \stackrel{\text{def}}{=} \int_0^5 f_0(x) \, \mathrm{d}x + \int_0^4 f_1(x) \, \mathrm{d}x + \int_0^3 f_2(x) \, \mathrm{d}x$$

Why studying metric graphs?
Physical motivations

Modeling structures where *only one spatial direction is important.*



A « fat graph » and the underlying metric graph

The differential system

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where the symbol $e \succ \mathrm{v}$ means that the sum ranges over all edges of vertex v and where $\frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v})$ is the outgoing derivative of u at v (*Kirchhoff's condition*).

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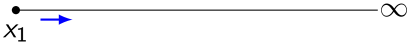
where the symbol $e \succ \mathrm{v}$ means that the sum ranges over all edges of vertex v and where $\frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v})$ is the outgoing derivative of u at v (*Kirchhoff's condition*).
We denote by $\mathcal{S}_\lambda(\mathcal{G})$ the set of nonzero solutions of the differential system.

Kirchhoff's condition: degree one nodes



$$\lim_{\substack{t \to 0 \\ t>0}} \frac{u(x_1+t)-u(x_1)}{t} = 0$$

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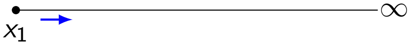
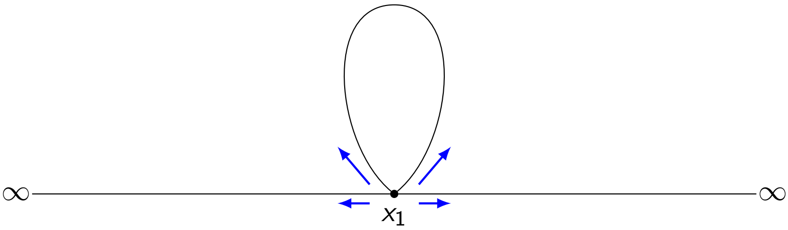


$$\lim_{\substack{t \to 0 \\ t>0}} \frac{u(x_1+t)-u(x_1)}{t} = 0$$

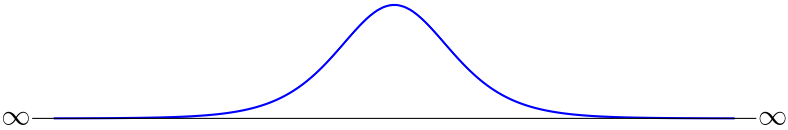
In other words, the derivative of u at x_1 vanishes: this is the usual Neumann condition.

Kirchhoff's condition in general: outgoing derivatives



$$\sum_{e \succ \mathrm{v}} \frac{\mathrm{d}u}{\mathrm{d}x_e}(\mathrm{v}) = 0$$

The real line: $\mathcal{G} = \mathbb{R}$



$$\mathcal{S}_\lambda(\mathbb{R}) = \left\{ \pm\varphi_\lambda(x+a) \;\middle|\; a \in \mathbb{R} \right\}$$

where the *soliton* φ_λ is the unique strictly positive and even solution to

$$u'' + |u|^{p-2}u = \lambda u.$$

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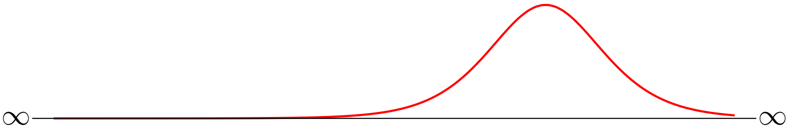

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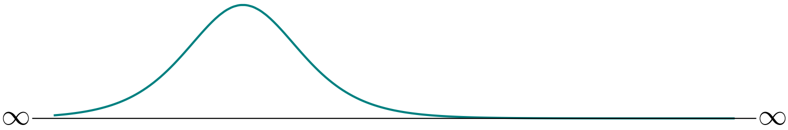

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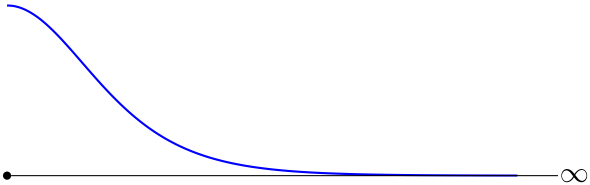
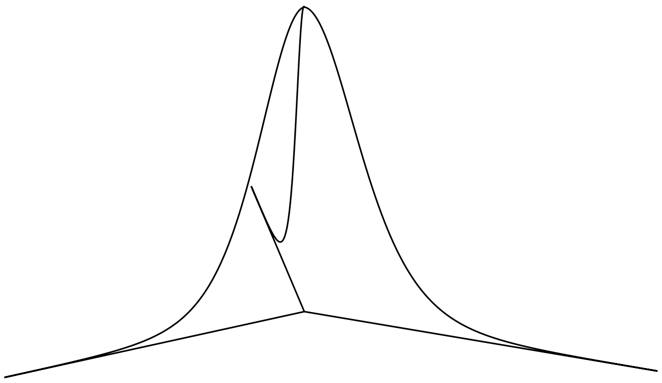
The halfline: $\mathcal{G} = \mathbb{R}^+ = [0, +\infty[$



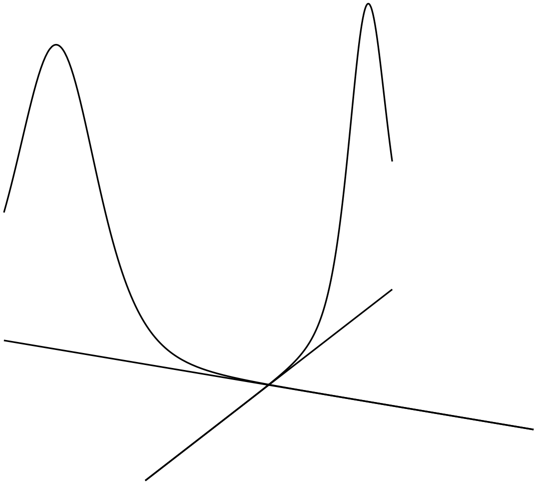
$$\mathcal{S}_\lambda(\mathbb{R}^+) = \left\{ \pm\varphi_\lambda(x)_{|\mathbb{R}^+} \right\}$$

Solutions are *half-solitons*: no more translations!

The positive solution on the 3-star graph



A continuous family of solutions on the 4-star graph



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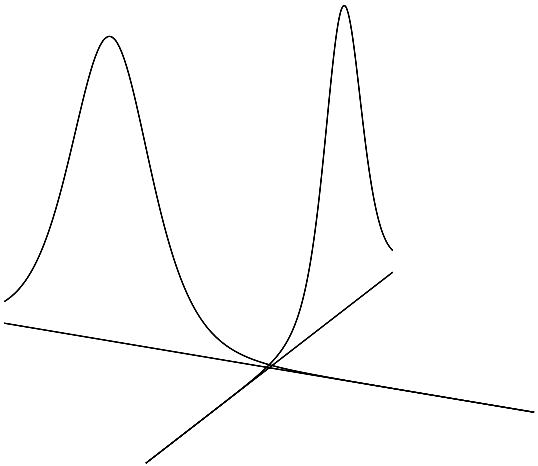

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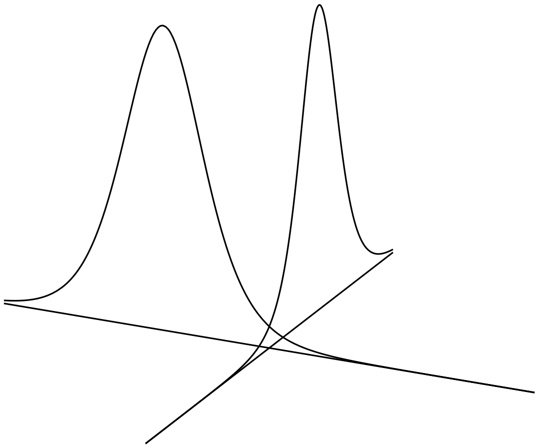

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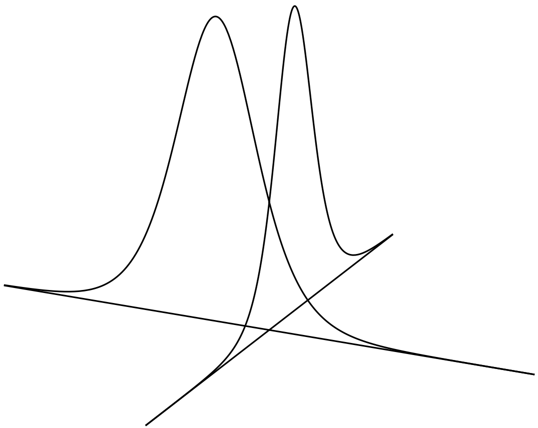

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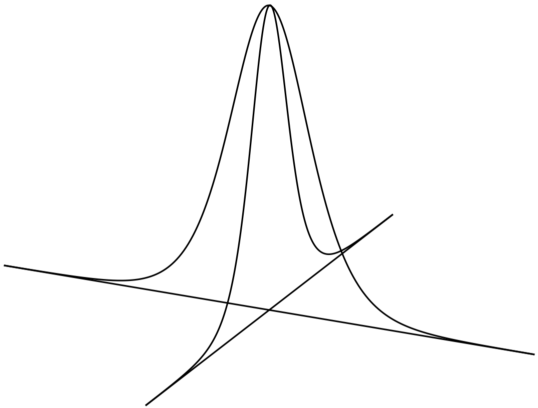

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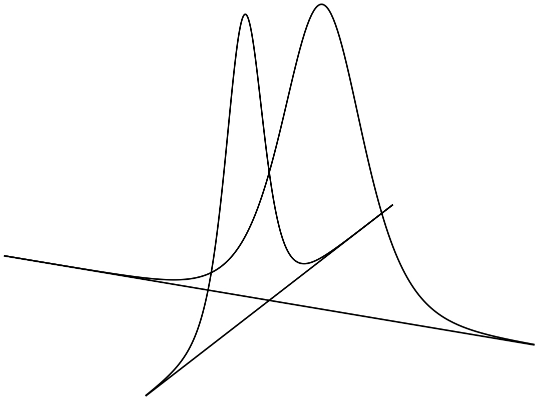

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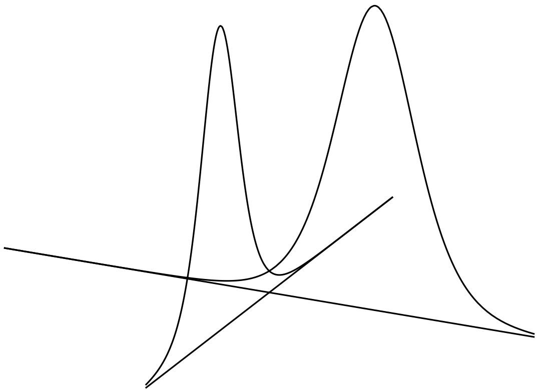

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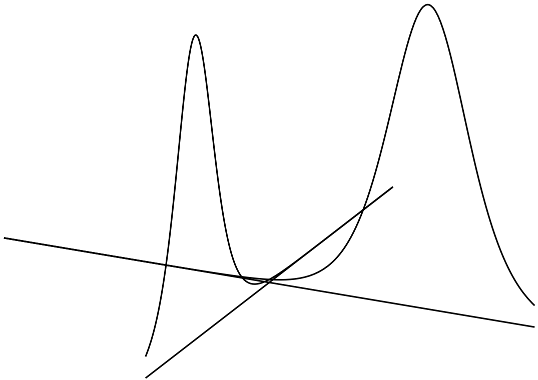

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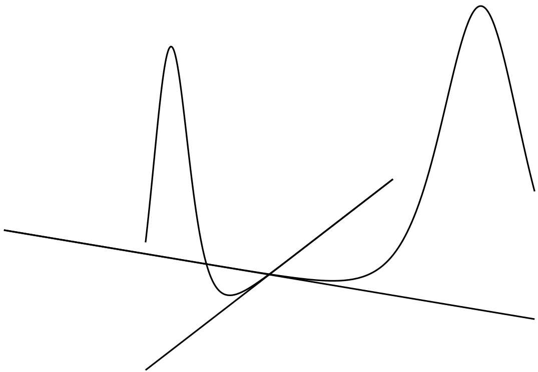

Variational formulation

We work on the Sobolev space

$$H^1(\mathcal{G}) := \Big\{u : \mathcal{G} \to \mathbb{R} \mid u \text{ is continuous}, u, u' \in L^2(\mathcal{G})\Big\}.$$

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Solutions of (NLS) correspond to critical points of the *action functional*

$$J_\lambda(u) := \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}.$$

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The level of the soliton φ_λ plays an important role in our analysis:

$$s_\lambda := J_\lambda(\varphi_\lambda).$$

The Nehari manifold

The functional J_λ is not bounded from below on $H^1(\mathcal{G})$, since if $u \neq 0$ then

$$J_\lambda(tu) = \frac{t^2}{2}\|u'\|^2_{L^2(\mathcal{G})} + \frac{\lambda t^2}{2}\|u\|^2_{L^2(\mathcal{G})} - \frac{t^p}{p}\|u\|^p_{L^p(\mathcal{G})} \xrightarrow[t\to\infty]{} -\infty.$$

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A common strategy is to introduce the *Nehari manifold* $\mathcal{N}_\lambda(\mathcal{G})$, defined by

$$\begin{aligned}\mathcal{N}_\lambda(\mathcal{G}) &:= \left\{u \in H^1(\mathcal{G}) \setminus \{0\} \mid J'_\lambda(u)[u] = 0\right\} \\ &= \left\{u \in H^1(\mathcal{G}) \setminus \{0\} \mid \|u'\|^2_{L^2(\mathcal{G})} + \lambda\|u\|^2_{L^2(\mathcal{G})} = \|u\|^p_{L^p(\mathcal{G})}\right\}.\end{aligned}$$

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If $u \in \mathcal{N}_\lambda(\mathcal{G})$, then

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{L^p(\mathcal{G})}^p.$$

In particular, J_λ is bounded from below on $\mathcal{N}_\lambda(\mathcal{G})$.

Two action levels

- « Ground state » action level:

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- *Minimal action solution*: solution $u \in \mathcal{S}_\lambda(\mathcal{G})$ of the differential system (NLS) of level $\sigma_\lambda(\mathcal{G})$.

Four cases

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A1) $c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and both infima are attained;

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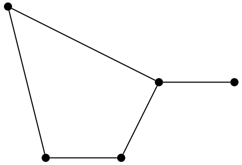
Theorem (De Coster, Dovetta, G., Serra (2023))

For every $p > 2$, every $\lambda > 0$, and every choice of alternative between A1, A2, B1, B2, there exists a metric graph $\mathcal{G}$ where this alternative occurs.

Case A1

$c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and both infima are attained



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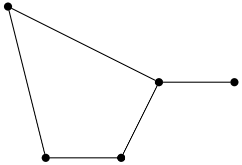

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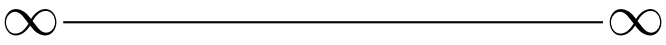

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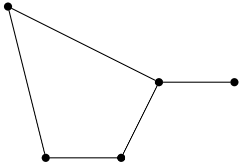

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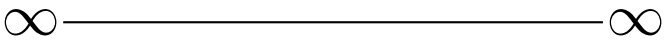

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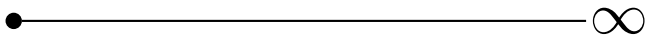

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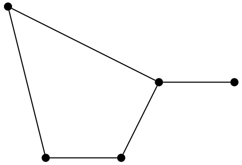

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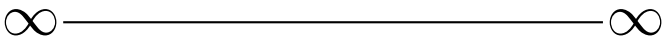

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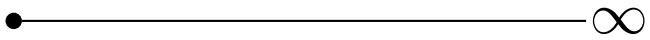

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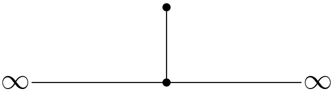
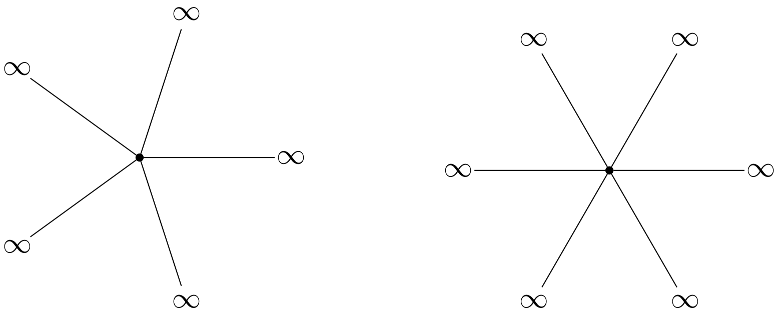


All graphs with $c_\lambda(\mathcal{G}) < s_\lambda$

Case B1

$c_\lambda(\mathcal{G}) < \sigma_\lambda(\mathcal{G})$, $\sigma_\lambda(\mathcal{G})$ is attained but not $c_\lambda(\mathcal{G})$

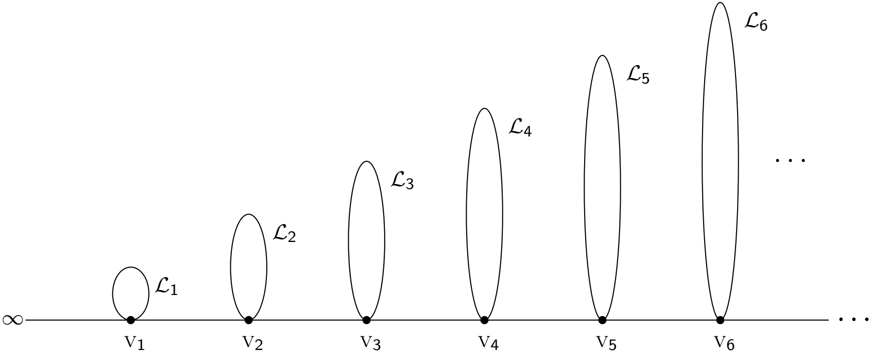


N-star graphs, $N \geq 3$

$$s_\lambda = c_\lambda(\mathcal{G}) < \sigma_\lambda(\mathcal{G}) = \frac{N}{2} s_\lambda$$

Case A2

$c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and neither infima is attained



$$s_\lambda = c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$$

Case B2

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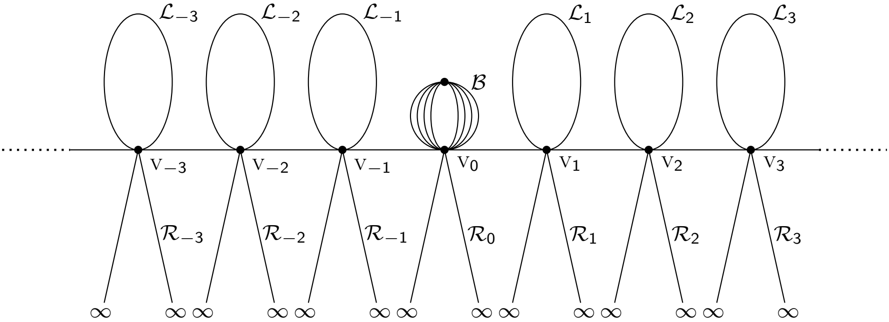

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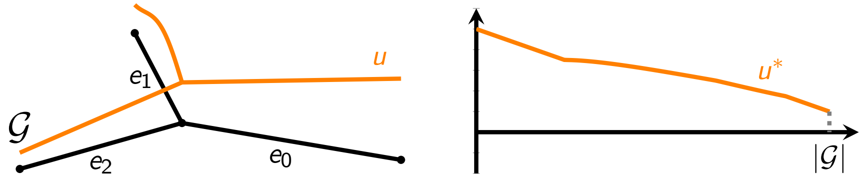
Decreasing rearrangement on the halfline



For all $1 \leq p \leq +\infty$,

$$\|u\|_{L^p(\mathcal{G})} = \|u^*\|_{L^p(0,|\mathcal{G}|)}.$$

The Pólya–Szegő inequality

Theorem

Let $u \in H^1(\mathcal{G})$ be a nonnegative function. Then its decreasing rearrangement u^* belongs to $H^1(0, |\mathcal{G}|)$, and one has

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \leq \|u'\|_{L^2(\mathcal{G})}.$$

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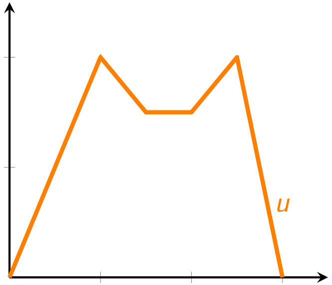
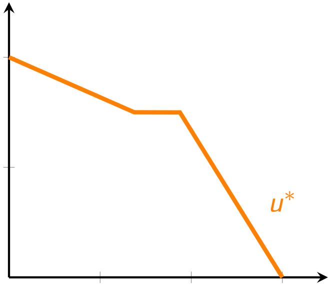
- Duff, G. *Integral Inequalities for Equimeasurable Rearrangements.* Canadian Journal of Mathematics **22** (1970), no. 2, 408–430.
- Friedlander, L. *Extremal properties of eigenvalues for a metric graph.* Ann. Inst. Fourier (Grenoble) **55** (2005) no. 1, 199–211.

The Pólya–Szegő inequality

A simple case: affine functions

We assume that u is piecewise affine.





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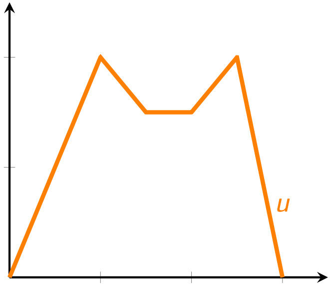

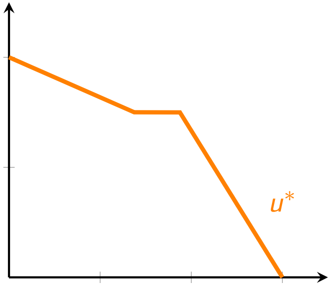

We consider a small open interval $I \subseteq u(\mathcal{G})$ so that $u^{-1}(I)$ consists of a disjoint union of open intervals on which u is affine.

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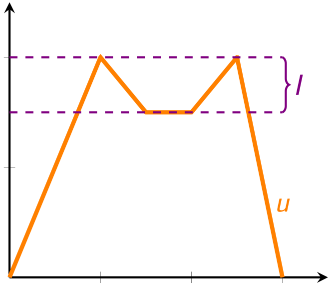

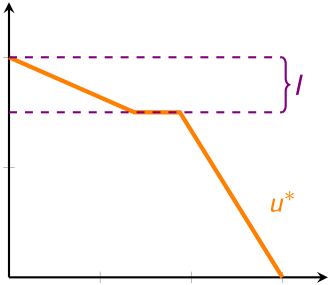

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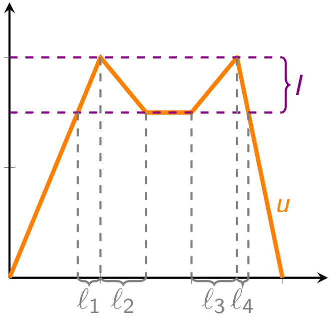

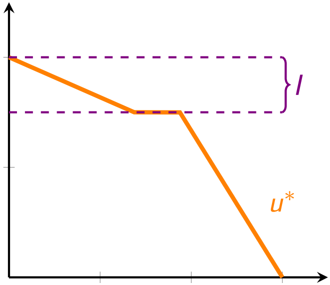

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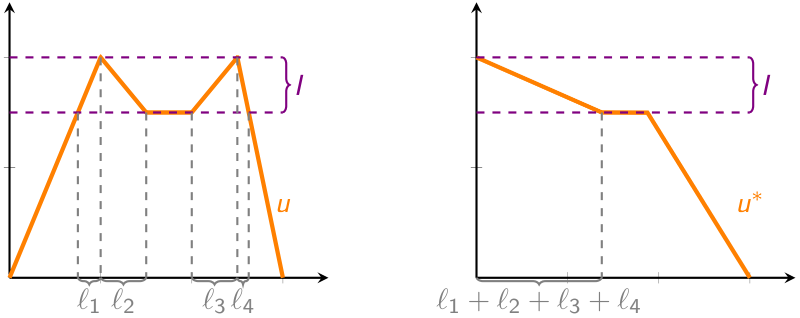

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A simple case: affine functions

Original contribution to $\|u'\|_{L^2}^2$:

$$A := \ell_1 \frac{|I|^2}{\ell_1^2} + \ell_2 \frac{|I|^2}{\ell_2^2} + \ell_3 \frac{|I|^2}{\ell_3^2} + \ell_4 \frac{|I|^2}{\ell_4^2}$$

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Contribution to $\|(u^*)'\|_{L^2}^2$:

$$B := \frac{|I|^2}{\ell_1 + \ell_2 + \ell_3 + \ell_4}$$

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$$\frac{\ell_1 + \ell_2 + \ell_3 + \ell_4}{4} \geq \frac{4}{\frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3} + \frac{1}{\ell_4}}$$

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A refined Pólya–Szegő inequality...

... or the importance of the number of preimages

Theorem

Let $u \in H^1(\mathcal{G})$ *be a nonnegative function. Let* $N \geq 1$ *be an integer. Assume that, for almost every* $t \in]0, \|u\|_\infty[$*, one has*

$$u^{-1}(\{t\}) = \{x \in \mathcal{G} \mid u(x) = t\} \geq N.$$

Then one has

$$\|(u^*)'\|_{L^2(0,|\mathcal{G}|)} \leq \frac{1}{N}\|u'\|_{L^2(\mathcal{G})}.$$

Assumption (H)

Definition (Adami, Serra, Tilli 2014)

We say that a metric graph $\mathcal{G}$ satisfies assumption (H) if, for every point $x_0 \in \mathcal{G}$, there exist two injective curves $\gamma_1, \gamma_2 : [0, +\infty[\to \mathcal{G}$ parameterized by arclength, with disjoint images except for an at most countable number of points, and such that $\gamma_1(0) = \gamma_2(0) = x_0$.

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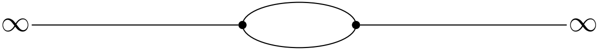

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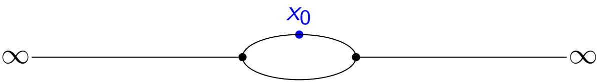

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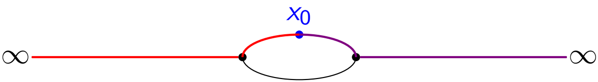

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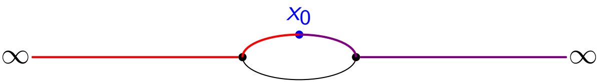

Consequence: *all* nonnegative $H^1(\mathcal{G})$ functions have at least two preimages for almost every $t \in]0, \|u\|_\infty[$.

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Mathematical motivations

Main message

Metric graphs allow to study interesting *one dimensional* problems and are much richer than the usual class of intervals of $\mathbb{R}$.

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Replacing $\mathcal{G}$ by noncompact smooth open sets $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$ and $H^1(\mathcal{G})$ by $H^1(\Omega)$ or $H^1_0(\Omega)$, one expects that the four cases A1, A2, B1, B2 actually occur.

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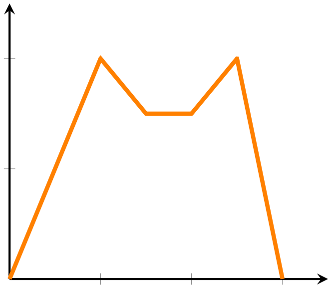
Thanks for your attention!



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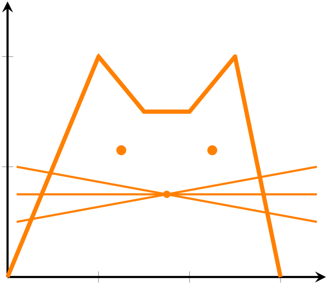

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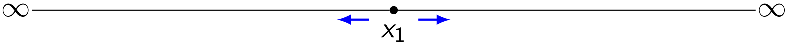
- De Coster C., Dovetta S., Galant D., Serra E. *On the notion of ground state for nonlinear Schrödinger equations on metric graphs.* Calc. Var. 62, 159 (2023).
- De Coster C., Dovetta S., Galant D., Serra E., Troestler C., *Constant sign and sign changing NLS ground states on noncompact metric graphs.* ArXiV preprint: `https://arxiv.org/abs/2306.12121`.

Overviews of the subject

- Adami R. *Ground states of the Nonlinear Schrodinger Equation on Graphs: an overview (Lisbon WADE).* https://www.youtube.com/watch?v=G-FcnRVvoos (2020)
- Adami R., Serra E., Tilli P. *Nonlinear dynamics on branched structures and networks.* https://arxiv.org/abs/1705.00529 (2017)
- Kairzhan A., Noja D., Pelinovsky D. *Standing waves on quantum graphs.* J. Phys. A: Math. Theor. 55 243001 (2022)

Kirchhoff's condition: degree two nodes



$$\left(\lim_{\substack{t \to 0 \\ t>0}} \frac{u(x_1 + t) - u(x_1)}{t} \right) + \left(\lim_{\substack{t \to 0 \\ t>0}} \frac{u(x_1 - t) - u(x_1)}{t} \right) = 0$$

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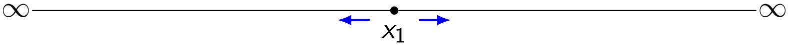

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In other words, the left and right derivatives of u are equal, which simply means that u is differentiable at x_1. This explains why usually we do not put degree two nodes.

A very useful tool: cutting solitons on halflines

Proposition

Assume that $\mathcal{G}$ has at least one halfline. Then,

$$c_\lambda(\mathcal{G}) \leq s_\lambda := J_\lambda(\varphi_\lambda)$$

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Proof.

□

Case A1

$c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and both infima are attained

Theorem (Adami, Serra, Tilli 2014)

Let $\mathcal{G}$ be a metric graph with finitely many edges, including at least one halfline. Assume that

$$c_\lambda(\mathcal{G}) < s_\lambda.$$

Then $c_\lambda(\mathcal{G})$ is attained, which means that there exists a ground state, so we are in case A1: $c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$, both attained.

Non-existence of ground states

Theorem (Adami, Serra, Tilli 2014)

If a metric graph $\mathcal{G}$ satisfies assumption (H), then

$$c_\lambda(\mathcal{G}) := \inf_{u \in \mathcal{N}_\lambda(\mathcal{G})} J_\lambda(u) = s_\lambda$$

but it is never achieved

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If a metric graph $\mathcal{G}$ satisfies assumption (H), then

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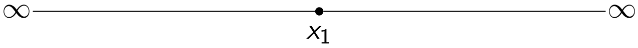
but it is never achieved, unless $\mathcal{G}$ is isometric to one of the exceptional graphs depicted in the next two slides.

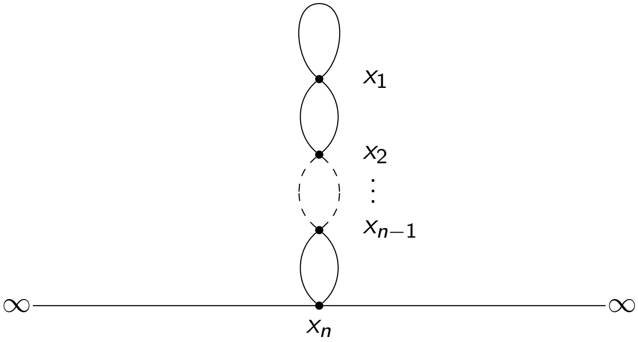
Non-existence of ground states

Exceptional graphs: the real line



Non-existence of ground states

Exceptional graphs: the real line with a tower of circles



A doubly constrained variational problem

We define

$$X_e := \left\{ u \in H^1(\mathcal{G}) \mid \|u\|_{L^\infty(\mathcal{G})} = \|u\|_{L^\infty(e)} \right\}$$

where e is a given bounded edge of $\mathcal{G}$

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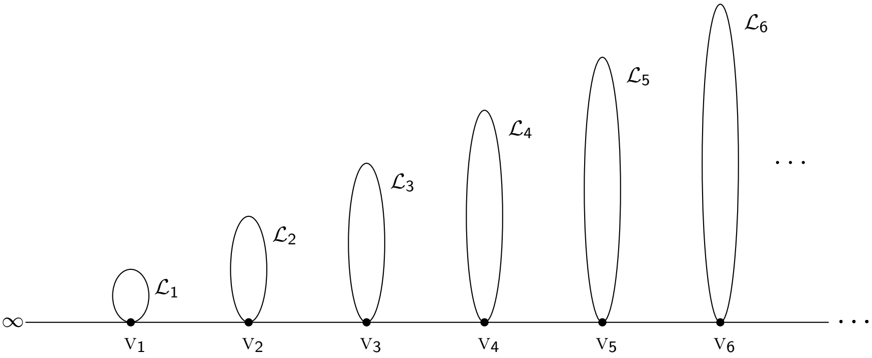
Theorem (De Coster, Dovetta, G., Serra (2023))

If $\mathcal{G}$ satisfies assumption (H) has a ***long enough*** *bounded edge e, then $c_\lambda(\mathcal{G}, e)$ is attained by a solution $u \in \mathcal{S}_\lambda(\mathcal{G})$, such that $u > 0$ or $u < 0$ on $\mathcal{G}$ and*

$$\|u\|_{L^\infty(e)} > \|u\|_{L^\infty(\mathcal{G} \setminus e)}.$$

What's going on in case A2?

$c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G})$ and neither infima is attained



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- Since $\mathcal{G}$ has at least one halfline and satisfies assumption (H), one has $c_\lambda(\mathcal{G}) = s_\lambda$ and the infimum is not attained (as $\mathcal{G}$ does not belong to the class of exceptional graphs).

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- According to the existence Theorems, $c_\lambda(\mathcal{G}, \mathcal{L}_n)$ is attained *by a solution of (NLS)* for every n large enough.
- One obtains

$$s_\lambda = c_\lambda(\mathcal{G}) \leq \sigma_\lambda(\mathcal{G}) \leq \liminf_{n\to\infty} c_\lambda(\mathcal{G}, \mathcal{L}_n) = s_\lambda,$$

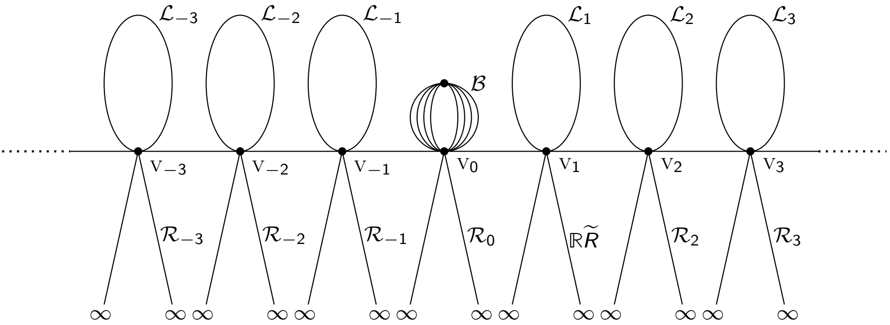
so

$$c_\lambda(\mathcal{G}) = \sigma_\lambda(\mathcal{G}) = s_\lambda$$

and neither infimum is attained.

What's going on in case B2?

$c_\lambda(\mathcal{G}) < \sigma_\lambda(\mathcal{G})$ and neither infima is attained

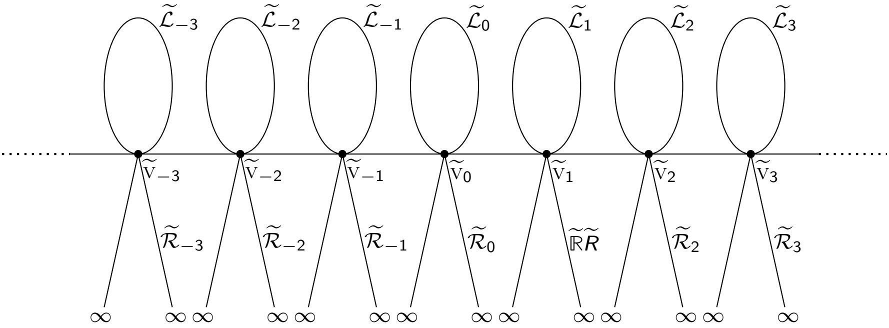


The graph $\mathcal{G}_N$.

The loops $\mathcal{L}_i$ have length N and $\mathcal{B}$ is made of N edges of length 1.

What's going on in case B2?

A second, periodic, graph



The graph $\widetilde{\mathcal{G}}_N$.

The loops $\widetilde{\mathcal{L}}_i$ have length N.

What's going on in case B2?

Two problems at infinity

- Since $\mathcal{G}_N$ and $\widetilde{\mathcal{G}}_N$ satisfy (H) and contain halflines, one has

$$s_\lambda = c_\lambda(\mathcal{G}_N) = c_\lambda(\widetilde{\mathcal{G}}_N),$$

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- One then shows, using suitable rearrangement techniques, that

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- One then shows, using suitable rearrangement techniques, that

$$\sigma_\lambda(\mathcal{G}_N) = \sigma_\lambda(\widetilde{\mathcal{G}}_N),$$

  but that $\sigma_\lambda(\mathcal{G}_N)$ *is not attained*.
- Therefore, for large N, we have that

$$s_\lambda = c_\lambda(\mathcal{G}_N) < \sigma_\lambda(\mathcal{G}_N),$$

  and neither infima is attained, as claimed.